DISSOLUTION OF AN ELLIPSOIDAL BUBBLE IN A SLIGHTLY VISCOUS LIQUID

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The steady-state velocity, the degree of deformation, and the convective-diffusion-limited rate of quasisteady-state growth (or dissolution) are considered for gas bubbles having shapes close to those of spheres or disks. It is assumed that there are no surface-active substances in the liquid. A qualitative agreement is found between the calculated dissolution rate and the experimental data.

NOTATION

a—radius of the sphere of equivalent volume;
u—bubble velocity with respect to the still liquid at infinity;
γ—kinematic viscosity of the liquid;
ρ—liquid density;
D—gas diffusion coefficient in the liquid;
σ—surface tension;
g—gravitational acceleration;
d—[$R = 2au/\nu$]—Reynolds number;
e—[$P = 2au/D$]—Peclet number;
f—[$W = 2a\rho u^2/\sigma$]—Weber number.

1. Shape and steady-state velocity of the gas bubble. The rate of quasi-steady-state dissolution of a gas bubble in a liquid has been calculated by Levich [1] for the following conditions: a) the bubble is of spherical shape; b) the velocity field of the liquid flow around the bubble is that of an ideal liquid. However, these two conditions may be incompatible, since the second implies that $R^{1/2} \gg 1$, so bubble deformation may have to be taken into account.

The question of bubble shape was discussed by Moore [2, 3], who showed that to determine the surface shape it is sufficient to calculate the pressure p in the flow of an ideal liquid around the bubble. Allowance for liquid viscosity leads to a correction of order 1/R in the pressure distribution near the bubble surface. Accordingly, for a given flow velocity around the bubble and for sufficiently large Reynolds numbers, the surface shape is given, with an accuracy to terms of order 1/R, by

$$p + \sigma(R_1^{-1} + R_2^{-1}) = p'. \tag{1.1}$$

Here R_1 and R_2 are the main radii of the curvature of surface, and p' is the gas pressure in the bubble.

The degree of bubble deformation depends on the Weber number W. When W < 2, the bubble shape is close to spherical. When W > 2, the bubble may be assumed to take on the shape of an oblate ellipsoid of revolution. We let χ be the ratio of the semimajor axis (perpendicular to the liquid flow) to the semiminor axis (parallel to the liquid flow). The $\chi(W)$ dependence was found in [3] from the condition that Eq. (1.1) holds at the point on the bubble farthest upstream and at a point at the intersection of the bubble surface with the horizontal plane of symmetry; at this latter point, it was shown that

$$\chi - 1 = {}^9/_{64}\, W \quad \text{for } \chi - 1 \ll 1, \tag{1.2}$$

$$W = \pi^2 \chi^{-1/3} (1 - 4/\pi\chi)^2 \quad \text{for } \chi^2 \gg 1. \tag{1.3}$$

The steady-state velocity of an ellipsoid was calculated by Moore [3] from the rate at which the liquid's kinetic energy is dissipated. With an accuracy to terms of the order of $R^{-1/2}$, his results were

$$u = {}^1/_9\, (ga^2/\nu)\, [1 - {}^4/_3 (\chi - 1)] \quad \text{for } \chi - 1 \ll 1, \tag{1.4}$$

$$u = {}^1/_6 (\pi g a^2 / \nu \chi^{7/3}) (1 - 4/\pi\chi)^2 \quad \text{for } \chi^2 \gg 1. \tag{1.5}$$

The first term in Eq. (1.4) is the same as that given by Levich [1]; using it, we may calculate

$$\chi - 1 = {}^{9}/_{32}\, a\rho u^2 / \sigma = 0.043\, (\rho / \sigma)\, g^{1/3} \nu^{4/3} R^{5/3} = $$
$$= 3.5 \cdot 10^{-3} \rho g^2 a^5 / \sigma \nu^2 . \qquad (1.6)$$

Substituting (1.6) into (1.4), we find that

$$u = {}^{1}/_{9}\, (ga^2 / \nu)\, (1 - 4.6 \cdot 10^{-3} \rho g^2 a^5 / \sigma \nu^2) \qquad (1.7)$$

The experimental data in [4] show that as an air bubble moves in water at 6° C, there is an abrupt increase in the drag coefficient at approximately R = 250; in water at 19° C, this increase starts at approximately R = 400. In the first case, we have $\rho = 1.00$ g/cm^3, $\nu = 0.0147$ cm^2/sec, and $\sigma = 0.75 \cdot 10^{-5}$ J/cm^2; in the second, $\rho = 1.00$ g/cm^3, $\nu = 0.0102$ cm^2/sec, and $\sigma = 0.73 \cdot 10^{-5}$ J/cm^2. Equation (1.6) yields $\chi - 1 = 4.5 \cdot 10^{-5}\, R^{5/2}$ for the water at 19° C. The effect of deformation on the flow velocity becomes significant when the right-hand side of Eq. (1.6) becomes approximately equal to 0.5, which corresponds to W ≈ 2.

When $\chi^2 \gg 1$, it follows from Eqs. (1.3) and (1.5) that

$$\chi^2 = \rho g a^3 u / 3\pi\sigma\nu . \qquad (1.8)$$

Equation (1.3) shows that an approximate expression for the steady-state ascent velocity can be found in the region $3 < \chi < 6$ under the assumption W = const. It was established in [4, 5] in an analogous manner, on the basis of experimental data, that the steady-state ascent velocity of a bubble of moderate size and having a shape close to that of an oblate ellipsoid can be found from the condition W = 3.65:

$$u = 1.35\, (\sigma / \rho a)^{1/2} . \qquad (1.9)$$

A larger bubble would have a mushroom shape. This indicates that the region occupied by the turbulent wake covers about half the bubble surface. The drag of such a bubble is proportional to the square of the relative velocity. The steady-state velocity of a mushroom-shaped bubble is [4]

$$u = 1.02\sqrt{ga} . \qquad (1.10)$$

Accordingly, the region in which Eq. (1.9) is applicable may be assumed bounded from above by $a = 1.3\, (\sigma/\rho a)^{1/2}$. For an air bubble in water, this means $a < 0.35$ cm. Accordingly, Eqs. (1.8) and (1.9) yield the following bubble deformation for the region $\chi^2 \gg 1$:

$$\chi^2 = 0.143\, (ga^2 / \nu)\, (\rho a / \sigma)^{1/2} . \qquad (1.11)$$

The right-hand side of this equation is on the order of the ratio of the velocity of an equivalent-volume sphere to that of a deformed ellipsoid.

The region of applicability of (1.11) is smaller than that for the relation (1.9) between the ascent velocity of an ellipsoidal bubble and its equivalent radius. As shown below, Eq. (1.11) holds only when the condition $\chi^2 \ll \sqrt{R}$ holds. Otherwise, the corrections associated with a more accurate account of the liquid motion in the hydrodynamic boundary layer and in the wake lead to a different dependence of the bubble ascent velocity on the degree of deformation, and thus to a different dependence of the degree of deformation on the equivalent radius.

Moreover, at large values of χ, the actual bubble shape given by Eq. (1.1) differs from ellipsoidal. As was shown in [3], Eq. (1.1) is satisfied over the entire bubble surface only approximately, with a maximum error of 10% at $\chi = 2$ or 55% at $\chi = 4$. Approximately the same results were obtained by Kiselev [6] in a different manner. However, a comparison of the results obtained by Moore [3] with the experimental data of Haberman and Morton [4] shows that the steady-state bubble velocity apparently does not depend significantly on the deviation of the actual surface shape from the ellipsoidal. The calculated ellipsoidal velocity coincides with the experimental bubble velocity with an error no greater than 20% in the region up to $\chi = 4$, while the ellipsoid velocity may differ by an order of magnitude from that of an equivalent-volume sphere.

2. Convective diffusion to the ellipsoidal surface. The dissolution rate of an ellipsoidal bubble should differ from that of a spherical one of equivalent volume because of the different steady-state velocity, surface area, and distribution of the effective thickness of the diffusion boundary layer along the surface.

We assume that the velocity field of the flow around the bubble is that of an ideal liquid. For a given velocity of the ellipsoidal bubble, the diffusion flux to its surface can be calculated in a manner similar to that proposed by Levich for the case of a spherical bubble [1].

We assume that the origin of coordinates is at the center of the ellipsoid, and that the z-axis is parallel to the liquid flow velocity u. The equation of the surface of an axisymmetric ellipsoid is

$$\frac{x^2+y^2}{l_x^2}+\frac{z^2}{l_z^2}=1 \quad \left(\frac{l_x}{l_z}=\chi>1\right). \tag{2.1}$$

We introduce an orthogonal coordinate system α, β, φ such that

$$x=k\,[(1+\alpha^2)\,(1-\beta^2)]^{1/2}\cos\varphi,\quad y=k\,[(1+\alpha^2)\,(1-\beta^2)]^{1/2}\sin\varphi \tag{2.2}$$
$$z=k\alpha\beta\,.$$

It follows that

$$dx^2+dy^2+dz^2=h_\alpha^2 d\alpha^2+h_\beta^2 d\beta^2+h_\varphi^2 d\varphi^2\,,$$
$$h_\alpha=k\left(\frac{\alpha^2+\beta^2}{1+\alpha^2}\right)^{1/2},\quad h_\beta=k\left(\frac{\alpha^2+\beta^2}{1-\beta^2}\right)^{1/2},\quad h_\varphi=k\,[(1+\alpha^2)(1-\beta^2)]^{1/2}. \tag{2.3}$$

The equation for the surface of the ellipsoid corresponds to $\alpha = \alpha_0$; then

$$k\,(1+\alpha_0^2)^{1/2}=l_x,\qquad k\alpha_0=l_z\,. \tag{2.4}$$

Lamb [7] has calculated the velocity potential Φ for the flow of an ideal liquid around an ellipsoid:

$$\Phi=uk\beta\,[\alpha+p\,(1-\alpha\,\mathrm{arcctg}\,\alpha)]\,,$$
$$p=[\mathrm{arcctg}\,\alpha_0-\alpha_0/(1+\alpha_0^2)]^{-1}\,. \tag{2.5}$$

Accordingly, the velocity components v_α, v_β in the ellipsoidal coordinate system are

$$v_\alpha=u\beta\left(\frac{1+\alpha^2}{\alpha^2+\beta^2}\right)^{1/2}\left[1-p\left(\mathrm{arcctg}\,\alpha-\frac{\alpha}{1+\alpha^2}\right)\right], \tag{2.6}$$
$$v_\beta=u\left(\frac{1-\beta^2}{\alpha^2+\beta^2}\right)^{1/2}[\alpha+p\,(1-\alpha\,\mathrm{arcctg}\,\alpha)]\,.$$

The convective-diffusion equation becomes

$$\frac{v_\alpha}{h_\alpha}\frac{\partial c}{\partial\alpha}+\frac{v_\beta}{h_\beta}\frac{\partial c}{\partial\beta}=\frac{D}{h_\alpha h_\beta h_\varphi}\left(\frac{\partial}{\partial\alpha}\frac{h_\beta h_\varphi}{h_\alpha}\frac{\partial c}{\partial\alpha}+\frac{\partial}{\partial\beta}\frac{h_\alpha h_\varphi}{h_\beta}\frac{\partial c}{\partial\beta}\right), \tag{2.7}$$

with boundary conditions

$$c=c_s \quad \text{for } \alpha=\alpha_0,\ \beta\neq-1\,,$$
$$c\to c_0 \quad \text{for } k\alpha\to\infty\,, \tag{2.8}$$
$$c=c_0 \quad \text{for } \beta=-1,\ \alpha\neq\alpha_0\,.$$

The variable α may be replaced by the new variable γ in the following manner:

$$\alpha=\alpha_0\,(1+\delta\gamma),\qquad \delta=(1+\alpha_0^2)\,\alpha_0^{-1}\,(D/ukp)^{1/2}.$$

The quantity δ^{-2} is on the order of the Peclet number. According to the conditions of this problem, we have $\delta \ll 1$.

With an accuracy to terms of the order of δ, the convective-diffusion equation may be written

$$2\beta\gamma\frac{\partial c}{\partial\gamma}+(1-\beta^2)\frac{\partial c}{\partial\beta}=\frac{\partial^2 c}{\partial\gamma^2}\,. \tag{2.9}$$

Using the Mises transformation from the variables γ, β to the new variables $\psi = \gamma(1 - \beta^2), \beta$, we can reduce Eq. (2.9) to an equation of the heat-conduction type:

$$\frac{\partial c}{\partial\tau}=\frac{\partial^2 c}{\partial\psi^2}\quad\left(\tau=\frac{2}{3}+\beta-\frac{\beta^3}{3}\right), \tag{2.10}$$

with the initial and boundary conditions

$$\begin{aligned} c &= c_s \quad \text{for } \psi=0,\ \tau\neq 0, \\ c &\to c_0 \quad \text{for } \psi\to\infty, \\ c &= c_0 \quad \text{for } \psi\neq 0,\ \tau=0. \end{aligned} \tag{2.11}$$

The solution of Eq. (2.10) with conditions (2.11) is

$$\begin{aligned} c - c_s &= (c_0 - c_s)\,\mathrm{erf}\,(\psi / 2\sqrt{\tau}), \\ \mathrm{erf}\,(x) &= \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt. \end{aligned} \tag{2.12}$$

The diffusion flux density is

$$j = -D\left(\frac{1}{h_\alpha}\frac{\partial c}{\partial \alpha}\right)_0 = -\frac{D(c_0-c_s)(1-\beta^2)}{k\alpha_0\delta\sqrt{\pi\tau}}\left(\frac{1+\alpha_0^2}{\alpha_0^2+\beta^2}\right)^{1/2}, \tag{2.13}$$

while the total flux to the bubble surface is

$$I = 2\pi\int_{-1}^{1} jk^2\sqrt{(\alpha_0^2+\beta^2)(1+\alpha_0^2)}\,d\beta = 8\left(\frac{\pi}{3}uk^3pD\right)^{1/2}(c_0-c_s). \tag{2.14}$$

For a slightly deformed bubble, i. e., one whose shape differs little from the spherical, the parameters in Eq. (2.14) satisfy the conditions

$$\alpha_0 \gg 1,\quad k^3\alpha_0\,(1+\alpha_0^2) = a^3,\quad p = {}^3/_2\,\alpha_0^3\,(1 + {}^6/_5\,\alpha_0^{-2}). \tag{2.15}$$

The equation for the total flux becomes

$$I = 8\,({}^1/_2\,\pi u a^3 D)^{1/2}\,(1 + {}^1/_{10}\alpha_0^{-2})\,(c_0 - c_s). \tag{2.16}$$

When corrections of order α_0^{-2} are discarded, the main term of this equation agrees with that in the Levich equation [1].

Using Eqs. (1.4) and (1.6), we may write

$$I = \frac{8}{3}\left(\frac{\pi g a^5 D}{2\nu}\right)^{1/2}\left(1 - 1.6\cdot 10^{-3}\,\frac{\rho g^2 a^5}{\sigma\nu^2}\right)(c_0 - c_s). \tag{2.17}$$

We note that as the equivalent radius a increases, the corrections to the steady-state flow velocity (2.17) become noticeable at lower values of a than the corresponding corrections in Eq. (1.7) for the steady-state bubble velocity.

In the case of a highly deformed bubble,

$$\alpha_0 \ll 1,\quad k^3\alpha_0 = a^3,\quad p = (2/\pi)\,(1 + 4\alpha_0/\pi), \tag{2.18}$$

the total diffusion flux is, with an accuracy to terms of order α_0^2,

$$I = 8\,({}^2/_3 u a^3 D / \alpha_0)^{1/2}\,(1 + 2\alpha_0/\pi)\,(c_0 - c_s). \tag{2.19}$$

Equation (2.19) shows that the flux to the surface of the highly deformed bubble may slightly exceed the flux to the surface of a sphere of the same volume and velocity. This is a plausible result, since deformation of the bubble would be accompanied by an increase in its surface area. The surface area of a disk-shaped bubble increases in proportion to $\chi^{2/3}$. A slightly slower increase in the diffusion flux, proportional to $\chi^{1/2}$, occurs because the different parts of the bubble surface are not equally accessible to the diffusion flux. Taking into account the dependence of the steady-state velocity of a highly deformed bubble on its dimensions (1.9), and relation (1.11) between the degree of deformation and the steady-state velocity, we write

$$I = 4.68\left(\frac{ga^2\sqrt{\sigma}}{\nu\sqrt{\rho a}}\right)^{1/4}\sqrt{a^3D}\left[1 + 1.7\left(\frac{\nu\sqrt{\sigma}}{ga^2\sqrt{\rho a}}\right)^{1/2}\right](c_0 - c_s). \tag{2.20}$$

Comparing the main term in this equation with the Levich equation of the growth of a spherical bubble of equivalent volume,

$$I_0 = 3.35\,(ga^5D/\nu)^{1/2}/(c_0 - c_S)\,, \tag{2.21}$$

we see that the dissolution (or growth) rate of a disk-shaped bubble is less than that of a spherical one. However, the ratio of the total diffusion fluxes, $I/I_0 = 0.86\chi^{-1/2}$, cannot be less than of order unity, since Eq. (2.20) holds only for degrees of deformation which are not too large ($\chi^2 \ll R^{1/2}$).

Accordingly, the Levich equation (2.21) may be assumed to give correct order-of-magnitude growth rates (or diffusion rates) for an ellipsoidal bubble. It should be noted that there is a partial compensation for the effects of the decreased ascent velocity and the increased bubble surface area on the total diffusion flux as the degree of deformation increases.

3. Effect of energy dissipation in the hydrodynamic boundary layer and in the wake on the steady-state bubble velocity and on its growth or dissolution rate. As Moore has shown [3], a more accurate account of energy dissipation in the hydrodynamic boundary layer and in the wake leads to the following equation for the steady-state bubble velocity:

$$\frac{u_1}{u} = 1 - \frac{(1+\alpha_0^2)\sqrt{3\nu}\,(I_1 - I_2 + I_3)}{9\sqrt{2\pi kup}\,[\alpha_0 + (1-\alpha_0^2)\,\mathrm{arcctg}\,\alpha_0]}\,. \tag{3.1}$$

Here u is the bubble ascent velocity, with an accuracy of order $R^{-1/2}$

$$I_1 = \int_{-1}^{1}(\alpha_0^2+\beta^2)\,d\beta\int_0^{x(\beta)}d\tau\int_0^{x(\beta)}\frac{S(\tau)\,S(\tau')\,d\tau'}{(2x-\tau-\tau')^{3/2}}\,,$$

$$I_2 = 12\alpha_0^2\int_{-1}^{1}\frac{d\beta}{\alpha_0^2+\beta^2}\int_0^{x(\beta)}\frac{S(\tau)\,d\tau}{\sqrt{x-\tau}}\,,\quad I_3 = \frac{3p^2}{2(1+\alpha_0^2)^2}\int_0^{8/9}d\tau\int_0^{8/9}\frac{S(\tau)\,S(\tau')\,d\tau'}{\sqrt{16/9-\tau-\tau'}}\,, \tag{3.2}$$

$$S(x) = 3\alpha_0^2\,[\alpha_0^2+\beta^2(x)]^{-2},\quad x = {}^2/_9\,(\beta+1)^2(2-\beta)\,. \tag{3.3}$$

The value $\beta = -1$ (x = 0) corresponds to the point on the bubble farthest upstream.

It follows from Eqs. (3.1)–(3.3) that the steady-state velocity of a spherical bubble is

$$u_1 = ({}^1/_9\,ga^2/\nu)\,(1 + 2.21\,R^{-1/2})\,. \tag{3.4}$$

This result had been obtained in [3, 8].

The dependence of the integrals in (3.2) on α_0 or on χ is extremely complicated, so Moore [3] carried out a numerical calculation of the steady-state velocity only in the χ region from 1 to 4. These results should evidently be supplemented by a study of the behavior of (3.1) in the region $\chi^2 \gg 1$ ($\alpha_0^2 \ll 1$), for at least an estimate of the region of applicability of the equations of the previous section.

As $\alpha_0 \to 0$, the function

$${}^1/_2\,\alpha_0 S = {}^3/_2\,\alpha_0^3\,[\alpha_0^2+\beta^2(x)]^{-2} \tag{3.5}$$

tends toward zero in the region where $\beta^2(x) \gg \alpha_0^2$, and tends towards infinity in the region where $\beta^2(x) \ll \alpha_0^2$. The behavior of this function as $\alpha_0 \to 0$ is not affected when $\beta(x)$ is replaced by an expression valid for $\beta(x) \ll 1$:

$$\beta(x) = {}^3/_2\,(x - {}^4/_9)\,. \tag{3.6}$$

Since

$$\frac{\alpha_0}{2}\int_{-\infty}^{\infty}S\,dx = \frac{3\alpha_0^3}{2}\int_{-\infty}^{\infty}\frac{dx}{[\alpha_0^2 + {}^9/_4\,(x-{}^4/_9)^2]^2} = 1\,, \tag{3.7}$$

then the right-hand side of (3.5) is, as $\alpha_0 \to 0$, one representation of the δ-function. Accordingly, when $\alpha_0^2 \ll 1$, we may assume that

$$S(x) = (2/\alpha_0)\,\delta(x - {}^4/_9)\,. \tag{3.8}$$

Then

$$I_1 = \frac{27}{2\alpha_0^2}\int_0^1 \frac{\sqrt{\beta}\,d\beta}{(3-\beta^2)^{3/2}} = \frac{2.54}{\alpha_0^2}, \qquad I_3 = \frac{2{,}59}{\alpha_0^2}. \tag{3.9}$$

An evaluation shows that the integral I_2 is of the order of $\alpha_0^{-1/2}$. An account of this term in Eq. (3.1) leads to corrections of order $\alpha_0^{3/2}$. With an accuracy to terms on the order of α_0 inclusively, the equation for the steady-state ascent velocity is

$$\frac{u_1}{u} = 1 - \frac{0.31}{\alpha_0^{11/6}}\left(\frac{\nu}{ua}\right)^{1/2}\left(1+\frac{2\alpha_0}{\pi}\right) = 1 - \frac{0.44\chi^{11/6}}{\sqrt{R}}\left(1+\frac{2}{\pi\chi}\right). \tag{3.10}$$

When $\chi = 4$, Eq. (3.10) leads to a coefficient of $R^{-1/2}$ which differs by 20% from that given in [3].

Accordingly, the correction to the bubble velocity due to energy dissipation in the hydrodynamic boundary layer and in the wake turns out to be important both during the motion of a relatively small spherical bubble (since in this case a quantity of the order of $R^{-1/2}$ is not negligible) and in the case of relatively large, highly deformed bubbles (since the coefficient of the small quantity $R^{-1/2}$ is roughly proportional to χ^2).

Accordingly, the region of applicability of (3.10) is limited by the condition $\chi^2 \ll R^{1/2}$. The error due to the replacement of $\chi^{11/6}$ by χ^2 evidently does affect the order of magnitude of the correction.

The dependence (1.9) of the ascent velocity of a deformed bubble on the radius of an equivalent-volume sphere is not an exact consequence of (1.5) and (1.3), but may be considered a generalization of the experimental data; accordingly, the corrections to this equation on the order of $\chi^2/R^{1/2}$ need not be taken into account.

A more accurate equation for the flow velocity leads to corrections on the order of $R^{-1/2}$ in Eq. (2.19) for the growth or dissolution rate of the bubble. However, terms of this order also arise during the solution of the diffusion problem with an account of the real velocity field in the hydrodynamic boundary layer. Since the Prandtl number satisfies $\nu/D \gg 1$, the thickness of the diffusion boundary layer may be assumed negligibly small in comparison with the thickness of the hydrodynamic boundary layer. Therefore, in solving the convective-diffusion equation, it is sufficient to consider the velocity field at the bubble surface. The only nonvanishing component of the liquid velocity at the bubble surface is [3]

$$v_\beta = \frac{up}{1+\alpha_0^2}\left(\frac{1-\beta^2}{\alpha_0^2+\beta^2}\right)^{1/2} - \frac{1}{\alpha_0}\left(\frac{2up\nu}{3\pi k}\right)^{1/2}\left(\frac{\alpha_0^2+\beta^2}{1-\beta^2}\right)^{1/2}\int_0^{x(\beta)}\frac{S(x')\,dx'}{\sqrt{x-x'}}. \tag{3.11}$$

Here the functions S(x) and $x(\beta)$ are given by Eqs. (3.3). The convective-diffusion equation leads to an equation of the heat-conduction type (2.10), when ψ and τ are replaced by the new variables

$$\psi_1 = \gamma\left[1-\beta^2 - \frac{1+\alpha_0^2}{\alpha_0}\left(\frac{2\nu}{3\pi upk}\right)^{1/2}(\alpha_0^2+\beta^2)\int_0^{x(\beta)}\frac{S(x')}{\sqrt{x-x'}}\,dx'\right],$$

$$\tau_1 = \frac{2}{3}+\beta-\frac{\beta^3}{3} - \frac{1+\alpha_0^2}{\alpha_0}\left(\frac{2\nu}{3\pi upk}\right)^{1/2}\int_{-1}^{\beta} d\beta'\,(\alpha_0^2+\beta'^2)\int_0^{x(\beta')}\frac{S(x')}{\sqrt{x-x'}}\,dx'. \tag{3.12}$$

Then the equation for the total flux to the bubble surface becomes

$$\begin{aligned} I_1 &= 4\,(\pi uk^3pD)^{1/2}(c_0-c_s)\times \\ &\times\left[\frac{4}{3} - \frac{1+\alpha_0^2}{\alpha_0}\left(\frac{2\nu}{3\pi u pk}\right)^{1/2}\int_{-1}^{1} d\beta\,(\alpha_0^2+\beta^2)\int_0^{x(\beta)}\frac{S(x')}{\sqrt{x-x'}}\,dx'\right]^{1/2} = \\ &= I\left[1 - \frac{1+\alpha_0^2}{4\alpha_0}\left(\frac{3\nu}{2\pi upk}\right)^{1/2}\int_{-1}^{1} d\beta\,(\alpha_0^2+\beta^2)\int_0^{x(\beta)}\frac{S(x')}{\sqrt{x-x'}}\,dx'\right]. \end{aligned} \tag{3.13}$$

Here I is given by Eq. (2.14).

For a spherical bubble, we have

$$I_1 = I\left[1 - \frac{4}{5}(3\sqrt{3} - 2)\left(\frac{\nu}{2\pi u a}\right)^{1/2}\right] = I(1 - 1.45R^{-1/2}). \quad (3.14)$$

Assuming that the flow velocity also incorporates corrections on the order of $R^{-1/2}$, we find

$$I_1 = 3.35\,(ga^5D/\nu)^{1/2}(1 - 0.35R^{-1/2})(c_0 - c_s). \quad (3.15)$$

As (3.15) shows, these corrections largely compensate for each other.

Accordingly, for the growth (or dissolution) rate of a bubble close to spherical in shape, we have

$$I = 3.35\left(\frac{ga^5D}{\nu}\right)^{1/2}\left[1 - 1.6\cdot 10^{-3}\frac{\rho g^2 a^5}{\sigma\nu} - 1.05\left(\frac{\nu^2}{ga^3}\right)^{1/2}\right](c_0 - c_s). \quad (3.16)$$

This equation takes into account the effect of a slight deformation on the diffusion flux and the corrections on the order of $R^{-1/2}$ to the flow velocity around the bubble.

In the case of a disk-shaped bubble, Eq. (3.13) leads to

$$I_1 = I\left(1 - \frac{3\sqrt{3}}{4\sqrt{R}}\alpha_0^{-11/6}\int_0^{1}\frac{\beta\sqrt{\beta}\,d\beta}{\sqrt{3-\beta^2}}\right) = I\left(1 - \frac{0.33}{\sqrt{R}}\alpha_0^{-11/6}\right). \quad (3.17)$$

where I is given by (2.20).

We note that the correction to the diffusion flux for a highly deformed bubble arises primarily at the trailing edge of the surface, where the streamlines differ significantly from those of an ideal liquid.

Accordingly, the quasi-steady-state growth rate of a disk-shaped bubble is

$$I_1 = 4.68\left(\frac{ga^2\sqrt{\sigma}}{\nu\sqrt{\rho a}}\right)^{1/4}\sqrt{a^3D}\left(1 + \frac{2}{\pi\chi} - 0{,}33\frac{\chi^{11/6}}{\sqrt{R}}\right)(c_0 - c_s). \quad (3.18)$$

With an error no greater than 30% of the correction for $R \approx 10^3$, we may replace $\chi^{11/6}$ by χ^2, since $\chi^2 \ll R^{1/2}$. Furthermore, using the $\chi(a)$ dependence (1.11), we find that

$$I_1 = 4.68\left(\frac{ga^2\sqrt{\sigma}}{\nu\sqrt{\rho a}}\right)^{1/4}\sqrt{a^3D}\left[1 + 1.7\left(\frac{\nu\sqrt{\sigma}}{ga^2\sqrt{\rho a}}\right)^{1/2} - 2.9\cdot 10^{-2}\,ga\left(\frac{a}{\nu}\right)^{1/2}\left(\frac{\rho a}{\sigma}\right)^{3/4}\right](c_0 - c_s). \quad (3.19)$$

Equation (3.19) shows that χ^2 increases more rapidly than $R^{1/2}$ as the radius of the equivalent sphere increases. Therefore, the region of applicability of (3.19) is limited by the condition

$$2.9\cdot 10^{-2}\,ga\,(a/\nu)^{1/2}(\rho a/\sigma)^{3/4} \ll 1\,.$$

For an air bubble in water at 19° C, this means that $a^{9/4} \ll 0.085$ or $a < 0.26$ cm. We note that the bubble becomes nonspherical when $a \approx 0.06$ cm.

4. Comparison with experiment. When terms of the order of $1/\chi$ and $\chi^2/R^{1/2}$ in Eq. (3.19) are neglected, the main term in Eq. (3.19) increases as $a^{15/8}$ as the equivalent radius increases. This result may be assumed to agree with the experimental data of Shabalin et al. [9] on the dissolution of carbon dioxide from nonspherical bubbles: these authors concluded that the dissolution rate was proportional to the square of the equivalent radius.

These results can be expressed as a relation among dimensionless criterial numbers. The dimensionless mass flux is given by the Nusselt number $N = I/2\pi Da\,(c_0 - c_s)$. For a nearly spherical bubble, it follows from Eqs. (2.16), (3.14), and (1.6) that

$$\frac{N}{\sqrt{P}} = 1.13\left(1 - \frac{1.45}{\sqrt{R}} + 8.6\cdot 10^{-3}\frac{\rho}{\sigma}g^{1/3}\nu^{4/3}R^{5/3}\right). \quad (4.1)$$

For a disk-shaped bubble, Eqs. (2.19), (3.17), and (1.11) lead to

$$N / \sqrt{P} = 2.31 \cdot 10^{-2} g^{1/2} (\rho / \sigma)^{3/2} \nu^2 R^{5/2} \times \\ \times [1 - 0.33 \cdot 10^{-3} g (\rho / \sigma)^3 \nu^4 R^{9/2}] + 0.47. \quad (4.2)$$

In the figure we compare the calculated dependence of $N/P^{1/2}$ on R with experimental data on the dissolution of carbon dioxide in an aqueous solution of glucose by Redfield and Houghton [10]. The first three curves correspond to Eq. (4.1). The viscosity of the liquid is not small enough to permit the use of Eq. (4.2). Both asymptotic relations (4.1) and (4.2) were used to plot the fourth curve. The corresponding experimental points were obtained by measurements at various levels in a column.

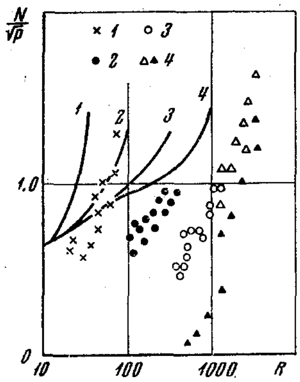

Fig. 1.

As has been noted several times before [10–12], the diffusion flux to a bubble surface at given volume, surface area, and velocity also depends on the time the bubble stays in the system. The reason for this phenomenon is not completely clear, since the time required to establish a steady-state diffusion flux during nondetached bubble flow should be on the order of a/u.

The occurrence of an explicit time dependence in the dependence of $N/P^{1/2}$ on R may be due to the adsorption of surface-active substances on the surface of the dissolving bubble; this would result in an expansion of the turbulent-wake region and in a turbulent transfer of the dissolved gas in the wake.

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